APPROXIMATELY FINITE ALGEBRAS

BY

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1. **Introduction.** In [5, Chapter IV], a study was made of certain types of rings of operators, the approximately finite factors of type II_1. These are factors of type II_1 which can be suitably approximated by finite dimensional subalgebras. Two major results were obtained. Firstly it was shown that on separable Hilbert space approximately finite factors exist and are all isomorphic (Theorem XIV), and secondly a nonapproximately finite factor was produced (Theorem XVI).

In this paper we shall extend the notion of approximate finiteness to factors on nonseparable Hilbert spaces, and obtain analogs of the results of [5]. In §6 we shall indicate how the discussion can be extended still further to finite AW*-algebras possessing a central trace.

2. **Various notions of approximate finiteness.** In [5, p. 759], a factor M of type II_1 on separable Hilbert space was defined to be approximately finite if there exists an increasing sequence $\{N_n\}$ of subfactors of M of type I with $\vee N_n = M$. (We shall use the symbol "$\vee$" to denote "ring generated by.") We would like to extend this definition to factors on not necessarily separable Hilbert spaces. We should not use exactly the same definition; for it is not hard to see that if $M = \vee N_n$ with $\{N_n\}$ an increasing sequence of subfactors of M of type I, then M has a faithful representation as operators on separable Hilbert space, so we would actually get nothing new. However, suppose $M = \vee N_\alpha$ where $\{N_\alpha\}$ is an ascending chain of subfactors of M approximately finite in the sense of [5]. Then it is possible that M does not have a faithful representation as operators·on separable Hilbert space, but we would still like to say that M is approximately finite (see [3, Theorem 2]). This leads to the following considerations.

Let M be a factor of type II_1 and $\mathcal{U}$ a collection of subrings of M. We shall call $\mathcal{U}$ *normal* if for any ascending chain $\{N_\alpha\}$ of members of $\mathcal{U}$ we have $\vee N_\alpha \in \mathcal{U}$. Since the intersection of normal collections is normal, we may speak of the normal collection generated by any collection of subrings of M. Let $\mathcal{U}(M)$ be the normal collection of subrings of M generated by the subfactors of type I. We shall say that M is *approximately finite* (A1) if $M \in \mathcal{U}(M)$.

It follows easily from the first statement of [3, Theorem 2], that $\mathcal{U}(M)$ must consist entirely of subfactors of M. In fact, $N \in \mathcal{U}(M)$ if and only if N is a subfactor of type I or one which is itself of type II_1 and approximately

finite (A1). Note that if M is a factor of type II_1 on separable Hilbert space, then M is approximately finite (A1) if and only if it is approximately finite in the sense of [5]. This remark also follows easily from [3, Theorem 2].

There is another way of extending the definition of approximate finiteness. Returning to the case of an approximately finite factor on separable Hilbert space, we have $M = \bigvee N_n$, $\{N_n\}$ being an increasing sequence of subfactors of M of type I. Define $P_1 = N_1$ and $P_n = N'_{n-1} \cap N_n$ for $n > 1$. Then P_n are mutually commuting subfactors of M of type I with $M = \bigvee P_n$. We are thus led to the following generalization: call M *approximately finite* (B) if $M = \bigvee P_\lambda$, the P_λ being mutually commuting subfactors of M of type I.

Finally we shall introduce another type of approximate finiteness, essentially a direct copy of Definition 4.3.1 of [5] .We shall call M *approximately finite* (A2) if given $A_i \in M$ $(i = 1, \cdots, n)$ and $\epsilon > 0$ we can find a subfactor N of M of type I containing elements B_i $(i = 1, \cdots, n)$ such that $[[A_i - B_i]] < \epsilon$. (Recall that for $A \in M$, $[[A]] = \mathrm{Tr}\,(A^*A)^{1/2}$, where Tr is the normalized trace on M.)

We shall prove below that approximate finiteness (A1) and (A2) are equivalent, and that approximate finiteness (B) is stronger than either of these[1]. A factor which is approximately finite (B) is determined, up to an algebraic isomorphism, by a single cardinal number.

3. **Equivalence of** (A1) **and** (A2). Any finite factor M is a metric space with metric $[[\ ]]$. Define $\chi(M)$ to be the density character of M relative to this metric, i.e., the smallest cardinal $\aleph$ such that there exists a subset of M of cardinality $\aleph$ which is $[[\ ]]$-dense in M. Now M is a pre-Hilbert space if we define $(A, B) = \mathrm{Tr}\,(B^*A)$, so M can be completed to a Hilbert space H. It is easily seen that if M is of type II (i.e., if H is infinite dimensional) then the dimension of H is just $\chi(M)$. Moreover M is faithfully represented as a ring of operators on $H[1]$. Thus, if $\chi(M) = \aleph_0$ then M is isomorphic to a ring of operators on separable Hilbert space.

LEMMA 1. *Assume M is approximately finite* (A2), *and let S be a subset of M of cardinality $\aleph$ ($\aleph$ infinite). Then there exists an $N \in \mathfrak{U}(M)$ such that $\chi(N) \leqq \aleph$ and $S \subset N$.*

Proof. We shall prove the result by a transfinite induction on $\aleph$. First we prove the result for $\aleph = \aleph_0$. Let $S = \{A_1, A_2, \cdots\}$, and construct factors N_n of type I as follows. By assumption there exists a factor N_1 of type I and $B_{11} \in N_1$ with $[[A_1 - B_{11}]] < 1$. Having N_{n-1}, apply Lemma 4.3.5 of [5] to find a factor of type I $N_n \supset N_{n-1}$ such that there exist $B_{n1}, \cdots, B_{nn} \in N_n$ with $[[A_i - B_{ni}]] < 1/n$ $(i = 1, \cdots, n)$. Let $N = \bigvee N_n$. Then $N \in \mathfrak{U}(M)$ and $\chi(N) = \aleph_0$. Now by [5, Theorem 1], $N = \bigvee N_n$ is the $[[\ ]]$-closure of $\bigcup N_n$. But since each A_i is in this closure, we have $A_i \in N$.

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[1] Examples of algebras approximately finite (A) but not (B) will be exhibited in a forthcoming paper.

Assume the result is proved for all cardinals $<\aleph$ ($\aleph > \aleph_0$), and let S be a subset of M of cardinality $\aleph$. If Ω is the first ordinal with cardinal $\aleph$ we may well-order S with ordinal Ω. Let A_α ($\alpha < \Omega$) be the elements of S ordered in this way. We shall construct an increasing chain $\{N_\alpha\}$ of members of $\mathfrak{U}(M)$ such that $A_\alpha \in N_\alpha$ and $\chi(N_\alpha) \leq \aleph_0 |\alpha|$. Here $|\alpha|$ denotes the cardinal of α. Assume we have $N_{\alpha'}$ for $\alpha' < \alpha$. If α is finite, $V_{\alpha' < \alpha} N_{\alpha'}$ has a dense set S' of cardinality $\aleph_0$. Using our induction hypothesis we can find an $N_\alpha \in \mathfrak{U}(M)$ such that $S' \cup \{A_\alpha\} \subset N_\alpha$ and $\chi(N_\alpha) \leq \aleph_0 = \aleph_0 |\alpha|$. If α is infinite, $V_{\alpha' < \alpha} N_{\alpha'}$ has a dense set S' of cardinality at most $|\alpha| \aleph_0 |\alpha| = |\alpha|$, and $|\alpha| < \aleph$. Therefore we can use the induction hypothesis to find an $N_\alpha \in \mathfrak{U}(M)$ such that $S' \cup \{A_\alpha\} \subset N_\alpha$ and $\chi(N_\alpha) \leq |\alpha| = \aleph_0 |\alpha|$. Thus in any case we can find $N_\alpha \in \mathfrak{U}(M)$ with $\chi(N_\alpha) \leq \aleph_0 |\alpha|$, such that $S' \cup \{A_\alpha\} \subset N_\alpha$, where S' is dense in $V_{\alpha' < \alpha} N_{\alpha'}$. But then $N_\alpha \supset (V_{\alpha' < \alpha} N_{\alpha'}) \cup \{A_\alpha\}$. This completes the construction of $\{N_\alpha\}$.

Now let $N = V_{\alpha < \Omega} N_\alpha$. Then $S \subset N$ and $\chi(N) \leq |\Omega| \aleph_0 |\Omega| = \aleph$, which completes the proof.

THEOREM 1. *Approximate finiteness* (A1) *and* (A2) *are equivalent.*

Proof. Assume M is approximately finite (A2). Then applying Lemma 1 with $S = M$, we can find $N \in \mathfrak{U}(M)$ such that $S \subset N$. But then $N = M$, so $M \in \mathfrak{U}(M)$ and M is approximately finite (A1).

Assume M is approximately finite (A1), and let $\mathfrak{V}$ be the collection of subfactors of M which are of type I or approximately finite (A2). Let $\{N_\alpha\}$ be an increasing chain of members of $\mathfrak{V}$, $N = V N_\alpha$. Assume $A_1, \cdots, A_n \in N$ and $\epsilon > 0$ are given.

Since N is the closure of $\cup N_\alpha$, we can find some $N_\alpha \in \{N_\alpha\}$ containing elements B_i ($i = 1, \cdots, n$) satisfying $[[A_i - B_i]] < \epsilon/2$. Since $N_\alpha \in \mathfrak{V}$, we can find a subfactor P of N_α of type I containing elements C_i ($i = 1, \cdots, n$) satisfying $[[B_i - C_i]] < \epsilon/2$. Then P is a type I subfactor of N containing elements C_i satisfying $[[A_i - C_i]] < \epsilon$. Thus $N \in \mathfrak{V}$. This shows that $\mathfrak{V}$ is normal. Since $\mathfrak{V}$ contains the subfactors of M of type I, we have $\mathfrak{V} \supset \mathfrak{U}(M)$. Since M is approximately finite (A1), we conclude $M \in \mathfrak{V}$, so M is approximately finite (A2).

Since approximate finiteness (A1) and (A2) are equivalent, we are justified in calling either of these *approximate finiteness* (A). Approximate finiteness (B) implies approximate finiteness (A): Assume M is approximately finite (B). If N and P are mutually commuting subfactors of M, both either of type I or approximately finite (A), then $N \vee P$ is either of type I or approximately finite (A). This follows easily from (A2). That M is approximately finite (A) is derived from this fact using transfinite induction.

4. An isomorphism theorem. We shall prove that factors approximately finite (B) are determined, to within isomorphism, by their density characters.

LEMMA 2. *Let M be approximately finite* (B): $M = V N_\lambda$, *the N_λ being mu-*

tually commuting subfactors of M of type I, *but not of type* I_1. *Then the number of N_λ's is just* $\chi(M)$.

Proof. Let R be the ring generated by the N_λ in the purely algebraic sense. Then if $\aleph$ is the number of N_λ's we see that R has a dense subset of cardinality $\leq \aleph \cdot \aleph_0 = \aleph$, since each N_λ has a dense subset of cardinality $\aleph_0$. Since R is dense in M, we have $\chi(M) \leq \aleph$.

Form the canonical Hilbert space H as in the beginning of §3. For each λ choose an $A_\lambda \in N_\lambda$ so that A_λ is not a scalar multiple of the identity. I claim that no A_μ is in the subspace (of H) spanned by $\{A_\lambda : \lambda \neq \mu\}$. For if it were, A_μ would be the $[[\ \ ]]$-limit of a net $\{A_i\}$ of elements of N_μ', so we would have $A_\mu \in N_\mu'$. Thus A_μ would be in the center of N_μ, contradicting our assumption that A_μ is not a multiple of the identity. Thus $\{A_\lambda\}$ is a subset of H of cardinality $\aleph$ such that no A_μ is in the subspace spanned by the remaining A_λ. It follows that the dimension of H is at least $\aleph$. Since this dimension is just $\chi(M)$, we have $\chi(M) \geq \aleph$.

THEOREM 2. *Let M and M_1 be approximately finite* (B) *with $\chi(M) = \chi(M_1)$.* *Then M and M_1 are algebraically isomorphic.*

Proof. Let $M = \vee N_\lambda$, N_λ being mutually commuting subfactors of type I none of which, we may assume, is of type I_1. Let $\{\lambda_i\}$ be any denumerably infinite set of λ's, and define $N = \vee N_{\lambda_i}$. Then N is a subfactor of M approximately finite (B), and $\chi(N) = \aleph_0$. Thus N is an approximately finite factor on separable Hilbert space. By [5, Lemma 4.4.3], we can find an increasing sequence $\{P_n\}$ of subfactors of N such that P_n is of type $I(2^n)$ and $\vee P_n = N$. Let $R_1 = P_1$ and $R_n = P_{n-1}' \cap P_n$ for $n > 1$. Then R_n are mutually commuting subfactors of N of type I_2 with $N = \vee R_n$.

By Lemma 2 the set $\{\lambda\}$ has cardinality $\aleph = \chi(M)$, so we may split it into $\aleph$ mutually disjoint subsets of cardinality $\aleph_0$. To each of these subsets apply the above construction to get factors of type I_2. Changing notation, we now have $M = \vee N_\lambda$, where N_λ are mutually commuting subfactors of type I_2. Now we can apply the same process to M_1 to get $M_1 = \vee P_\lambda$, where P_λ are mutually commuting subfactors of M_1. We may use the same indexing set $\{\lambda\}$ in both cases since there are just $\aleph$ subfactors in both cases.

Since N_λ and P_λ are both factors of type I_2 we can find an isomorphism onto $\phi : N_\lambda \to P_\lambda$. We use the symbol "$\phi$" for all the λ's since the intersection of any two N_λ's is just the complex numbers, where all the ϕ's agree. We shall extend the various ϕ's to a single isomorphism between M and M_1. Let N and P be the rings generated (in the purely algebraic sense) by $\{N_\lambda\}$ and $\{P_\lambda\}$ respectively. Now any $A \in N$ has the form $A = \sum_{i=1}^{k} A_i^1 \cdots A_i^n$ with $A_i^j \in N_{\lambda_j}$. For such an A define

$$\psi(A) = \sum_{i=1}^{k} \phi(A_i^1) \cdots \phi(A_i^n) \in P.$$

Then ψ is just the tensor product of the various ϕ's. It follows easily from the uniqueness of the trace that ψ is a trace-preserving isomorphism. Since N is dense in M and P is dense in M_1, ψ can be extended to an isomorphism between M and M_1.

5. **Examples.** In [5, Theorem XIII], a nonconstructive proof was given of the existence of approximately finite factors. We can given an equally nonconstructive proof for the existence of factors approximately finite (A).

THEOREM 3. *Any factor of type* II_1 *contains a maximal subfactor approximately finite* (A).

Proof. Let M be the given factor of type II_1, and consider the collection $\mathfrak{U}(M)$. By the very definition of $\mathfrak{U}(M)$, we can apply Zorn's lemma to find an $N \in \mathfrak{U}(M)$ maximal relative to the partial order of inclusion. Since $N \in \mathfrak{U}(N)$, we shall be through if we can show N is of type II. But if N is of type I we can apply Lemma 4.4.2 of [5] to obtain a member of $\mathfrak{U}(M)$ properly containing N. Thus N is of type II.

By the method of [5, Chapter V], it is easy to construct factors approximately finite (B). Let G be a countable group satisfying the conditions of Lemma 5.2.2 and 5.3.4 of [5]. For any infinite cardinal $\aleph$, let G_0 be the weak direct product of $\aleph$ copies of G, and let I_0' be the factor obtained from G_0 by the method of §5.3 of [5]. Corresponding to the decomposition of G_0 into mutually commuting subgroups isomorphic to G, there is a decomposition of I_0' into mutually commuting subfactors isomorphic to I' (where I' is the factor obtained from G). Since I' is approximately finite with density character $\aleph_0$, I_0' is approximately finite (B) with density character $\aleph$. We have proved,

THEOREM 4. *Given any infinite cardinal* $\aleph$, *there exists a factor* M *approximately finite* (B) *with* $\chi(M) = \aleph$.

Nonapproximately finite factors were produced in [5, Chapter IV]. Using these factors, we can produce factors of type II_1 with arbitrary density characters which are not approximately finite. Rather than go through this, however, we shall show that certain other factors of type II_1, which arise in a very natural way, are not approximately finite.

We shall be concerned with certain factors considered by Wright [8]. Let N be a finite AW*-algebra of type I (see [4, p. 236]), with homogeneous direct summands N_n, N_n being of type $I(p_n)$ with $p_1 < p_2 < \cdots$. If the maximal ideal space of the center of N_n is X_n, then the maximal ideal space X of the center of N can be identified with the Stone-Čech compactification of the disjoint union $Y = \bigcup X_n$. Now every $t \in X$ defines a maximal ideal M_t of $N : M_t = [A \in N : \mathrm{Tr}\,(A^*A)(t) = 0]$, where Tr is the central trace on N. The theorem of Wright that will concern us is [8, Theorem 5.1], namely that $t \in X - Y$ implies that N/M_t is an AW* factor of type II_1 with trace. It is shown in [1]

that N/M_t is then a W* algebra, i.e., it has a faithful representation as a ring of operators. Moreover it is shown in [2] that any faithful representation of N/M_t as a ring of operators must be on a nonseparable Hilbert space. We shall show here that N/M_t is not approximately finite (A).

First we prove a few lemmas. We shall write $M = N/M_t$. The first lemma shows that a type I_p subfactor of M can almost be lifted to a type I_p subalgebra of N. We shall denote the center of N_n by Z_n, and for the given integer p, $N_p = [n : p \text{ divides } p_n]$. Note that N is the direct sum of the N_n, so any element of N has the form $\{A_n\}$ with $A_n \in N_n$.

LEMMA 3. *Let P be a subfactor of M of type I_p, and let $\epsilon > 0$ be given. Then we can find a unitary $\overline{U} \in M$ with $[[\overline{U} - I]] < \epsilon$, and for each $n \in N_p$ a subalgebra P_n of N_n of type I_p with center Z_n, such that each $\overline{A} \in \overline{U} P \overline{U}^*$ can be lifted to an element of the form $\{A_n\} \in N$ with $A_n \in P_n$ whenever $n \in N_p$.*

Proof. For each n, find a projection $E_n \in N_n$ with $D(E_n) = [p_n/p] p/p_n$, where D is the normalized dimension function from the projections of N_n to Z_n; let $N' = \sum E_n N_n E_n$, the direct sum of the $E_n N_n E_n$. Since $D(E_n) \to 1$ (note that $p_n \to \infty$), every element of M can be lifted to one of N', so $M = N'/M_t \cap N'$.

Since, except for the finitely many n such that $p_n < p$, $E_n N_n E_n$ contains a set of p mutually equivalent orthogonal projections with supremum E_n, we can find a subalgebra Q_n of $E_n N_n E_n$ of type I_p with center $E_n Z_n$; for $p_n < p$ set $Q_n = \{0\}$. By [8, Theorem 5.1], $\sum Q_n/M_t \cap \sum Q_n$ is a subfactor of M of type I_p. Since any two subfactors of M of type I_p are unitarily equivalent, we can find a unitary $\overline{V} \in M$ such that $P = \overline{V}(\sum Q_n/M_t \cap \sum Q_n) \overline{V}^*$. By the spectral theorem we can find a unitary $\overline{W} \in M$ with $\|\overline{W} - \overline{V}\| < \epsilon$ (and so certainly with $[[\overline{W} - \overline{V}]] < \epsilon$) such that the spectrum of $\overline{W}$ is not the entire unit circle.

I claim that $\overline{W}$ can be lifted to a unitary of N'. In fact, if z is not in the spectrum of $\overline{W}$ then $\overline{A} = -i(\overline{W} + z)(\overline{W} - z)^{-1}$ is a self-adjoint element of M, and so can be lifted to a self-adjoint element $\{A_n\}$ of N'. Then if $W_n = z(A_n - i)(A_n + i)^{-1}$, the element $\{W_n\}$ of N' is unitary and maps into $\overline{W}$ under the natural map $N' \to M$.

Let $\overline{U} = \overline{W} \overline{V}^*$, $P_n = W_n Q_n W_n^*$. Then $\overline{U}$ is unitary with $[[\overline{U} - I]] < \epsilon$. Also, for $n \in N_p$ we have $E_n = I$, so P_n is of type I_p with center Z_n for such n. Now $\overline{U} P \overline{U}^* = \overline{W}(\sum Q_n/M_t \cap \sum Q_n) \overline{W}^* = \sum P_n/M_t \cap \sum P_n$. The result is therefore established.

The above lemma might be vacuous: it is possible that no p_n is divisible by p. The following lemma shows that this situation can be circumvented.

LEMMA 4. *We can find for each n a subalgebra N_n' of N_n such that,*
(1) *N_n' is of type $I(q_n)$, where q_n is divisible by n,*
(2) *the algebras M and $\sum N_n'/M_t \cap \sum N_n'$ are isomorphic.*

Proof. Let $q_n = n[p_n/n]$; find a projection $E_n \in N_n$ with $D(E_n) = q_n/p_n$ and set $N_n' = E_n N_n E_n$. Then N_n' is of type $\mathrm{I}(q_n)$ with center $E_n Z_n$ and (1) is immediate.

There is a natural *-homomorphism $\phi \colon \sum N_n' \to M$. We shall show that ϕ is onto and has kernel $M_t \cap \sum N_n'$, which will complete the proof. That ϕ is onto is the assertion that every element of M can be lifted to one of $\sum N_n'$, and this follows from the fact that $D(E_n) \to 1$. That the kernel of ϕ is $M_t \cap \sum N_n'$ is immediate.

The next lemma, the really crucial one, follows immediately from the main result of [6] and the fact that an algebra of type I_p is just the $p \times p$ matrix algebra over the center of the original algebra.

LEMMA 5. *There exists an $\epsilon > 0$ with the following property: for any $m > 0$ and $p > 1$, and algebra N of type I_{pm} with center Z, there exists an $A \in N$ with $\|A\| \leq 1$ such that if N' is a type I_m subalgebra of N with center Z then $[[B-A]] > \epsilon$ for all $B \in N'$.*

THEOREM 5. *The algebra M is not approximately finite* (A).

Proof. By Lemma 4 we may assume each p_n is divisible by n; we shall begin with $n = 2$. Apply Lemma 5 to each $N_{n!}$, with $p = n$ and $m = (n-1)!$, i.e. find $A_{n!} \in N_{n!}$ with $\|A_{n!}\| \leq 1$ such that if N' is a subalgebra of type I_p of $N_{n!}$ with center Z_n, and $p \leq n-1$, then $[[B - A_{n!}]] > \epsilon$ for all $B \in N'$.

If k is not of the form $n!$, define $A_k = 0$. Let $\overline{A}$ be the element of M corresponding to $\{A_n\} \in N$. We shall show that no type I subfactor of M can contain an element approximating $\overline{A}$ to within $\epsilon/2$. In fact, let P be a type I subfactor of M, say of type I_p, and let $\overline{B} \in P$. We may apply Lemma 3 to find a unitary $\overline{U} \in M$ with $[[\overline{U} - I]] < \epsilon \|\overline{B}\|^{-1}/4$ and a subalgebra P_n of N_n with the stated properties. Note that N_p contains all integers of the form $k!$ with $k \geq p$. Thus $\overline{U}\,\overline{B}\,\overline{U}^*$ can be lifted to an element $\{B_n\}$ with $B_n \in P_n$ for n of the form $k!$ $(k \geq p)$. But we know that for $n = k!$ with $k > p$ we have $[[B_n - A_n]] > \epsilon$. Thus $[[B_n - A_n]] > \epsilon$ for infinitely many n, so $[[\overline{U}\,\overline{B}\,\overline{U}^* - \overline{A}]] > \epsilon$. Now $[[\overline{B} - \overline{U}\,\overline{B}\,\overline{U}^*]] \leq 2\|B\| [[\overline{U} - I]] < \epsilon/2$. Therefore $[[\overline{B} - \overline{A}]] > \epsilon/2$. Since this holds for every element $\overline{B}$ of every type I subfactor of M, M cannot be approximately finite (A).

6. **Approximately finite AW*-algebras.** A study of the work above will reveal the fact that in studying approximately finite factors no use whatever was made of the underlying Hilbert space; the theory is entirely nonspatial. We could just as well have worked with a finite AW*-factor with trace. (For the basic theory of AW*-algebras we refer the reader to [4].) It happens that such an algebra has a faithful representation as a ring of operators on some Hilbert space [1], but this Hilbert space is not necessary for considerations of approximate finiteness. This suggests that the theory of approximately finite factors can be extended to arbitrary AW*-algebras of type II_1 with

central trace—the generalization here being from the factor case to the case of general center. It turns out that this extension can be made, and that the results obtained parallel almost completely those obtained above. In this section we shall merely state the results in the more general situation.

We start with an AW*-algebra M of type II_1 with center Z, M possessing a central trace Tr. For $A \in M$, $[[A]] = \text{Tr} (A^*A)^{1/2}$. We define "normal collection" as before, except that now the symbol "$\vee$" means "AW-subalgebra generated by;" and $\mathfrak{U}(M)$ is the normal collection of AW*-subalgebras of M generated by the AW*-subalgebras of type I with center Z. The definitions of approximately finiteness now read:

(A1) $M \in \mathfrak{U}(M)$,

(A2) given $A_1, \cdots, A_n \in M$ and $\epsilon > 0$ we can find an AW*-subalgebra N of M of type I with center Z containing elements B_i such that $[[A_i - B_i]] \leq \epsilon$ $(i = 1, \cdots, n)$,

(B) We can find P_λ, mutually commuting AW*-subalgebras of M of type I with center Z, such that $M = \vee P_\lambda$.

We define $\chi_1(M)$ to be the smallest cardinal $\aleph$ such that there exists a subset S of M of cardinality $\aleph$ with $S \vee Z = M$. It is seen that in the factor case, $\chi_1(M)$ need not equal the $\chi(M)$ defined in §3. In this case the inequalities $\chi_1(M) \leq \chi(M) \leq \aleph_0 \cdot \chi_1(M)$ are easily seen to hold.

The analogue of Theorem 1 is,

THEOREM 6. *Approximate finiteness* (A1) *and* (A2) *are equivalent.*

As before we call an algebra approximately finite (A) if either (A1) or (A2) holds. We have,

THEOREM 7. *Approximate finiteness* (B) *implies approximate finiteness* (A). *If* M *is approximately finite* (A) *and* $\chi_1(M) \leq \aleph_0$ *then* M *is approximately finite* (B).

Using the central trace on M, we can construct an AW*-module over Z in a natural way. For $A, B \in M$ define $(A, B) = \text{Tr} (B^*A)$. Then M is an inner product space over Z (see [7, §2]). By forming the AW*-completion $H(M)$ of M we obtain an AW*-module over Z. The analogue of Theorem 2 is,

THEOREM 8. *Let* M *and* M' *be approximately finite* (B) *with centers* Z *and* Z' *respectively. Assume* Z *is locally countably decomposable. Then* M *and* M' *are isomorphic if and only if* $H(M)$ *and* $H(M')$ *are isomorphic. If* $H(M)$ *and* $H(M')$ *are isomorphic, and* ϕ *is the isomorphism between* Z *and* Z' *inherent in the isomorphism between* $H(M)$ *and* $H(M')$, *the isomorphism between* M *and* M' *can be chosen to be an extension of* ϕ.

Theorem 3 goes over almost verbatim.

THEOREM 9. *Any* AW^*-*algebra of type* II_1 *with center* Z *which possesses a*

central trace contains a maximal AW-subalgebra with center **Z** which is approximately finite* (A).

In the factor case we saw that an algebra approximately finite (B) is determined by an infinite cardinal number, and furthermore that there is such an algebra for each cardinal number (Theorems 2 and 4). In the case of general (or almost general) center an algebra approximately finite (B) is determined by an AW*-module (Theorem 8). Not every AW*-module can arise in this way. Call an AW*-module **H** *purely infinite* if the algebra of operators on **H** is purely infinite [4, p. 240]. It is seen that the AW*-modules **H(M)** are always purely infinite.

THEOREM 10. *Let **H** be a purely infinite* AW*-*module over the commutative AW*-algebra **Z**. Then there exists an AW*-algebra **M** with center **Z** which is approximately finite* (B) *and such that **H(M)** is isomorphic to **H**.*

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